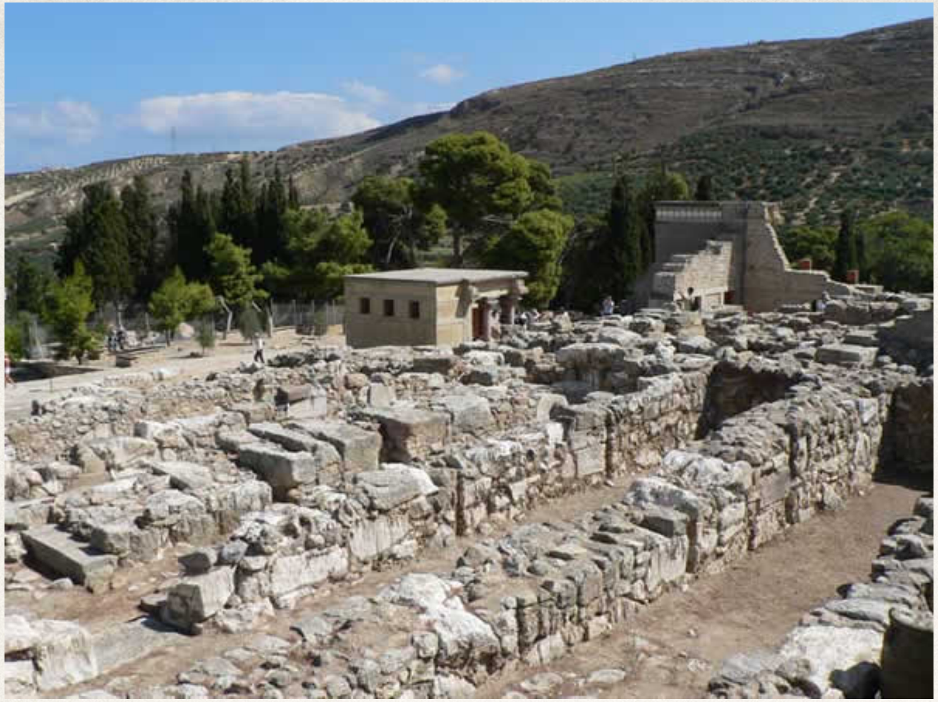

Set theory, univalent foundations and their relative positions in the foundations of mathematics

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In memoriam Vladimir Voevodsky

04/06/1966 -30/09/2017

- Founder of HOTT and inventor of univalent foundations

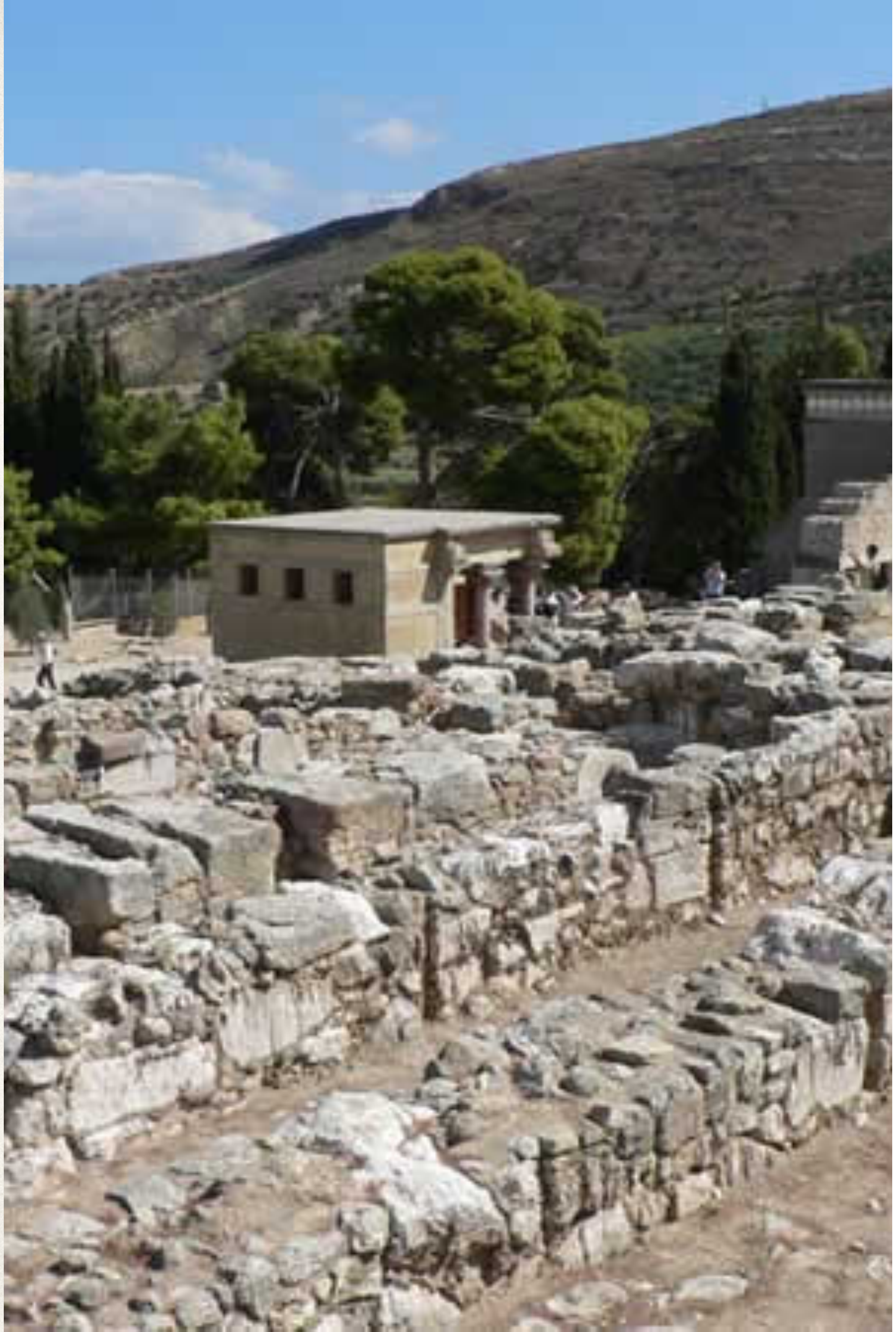
The context

- Ever since late 19th century most people have considered set theory as THE foundation of mathematics.
- In 2006 homotopy type theory was born and it resulted in univalent foundations, which are different than the set-theoretic foundations. In some respects the two foundations appear to contradict each other.
- Our topic: is this really the case? We answer negatively.

Set theory, an old and proven candidate

Are these foundations or their ruins?

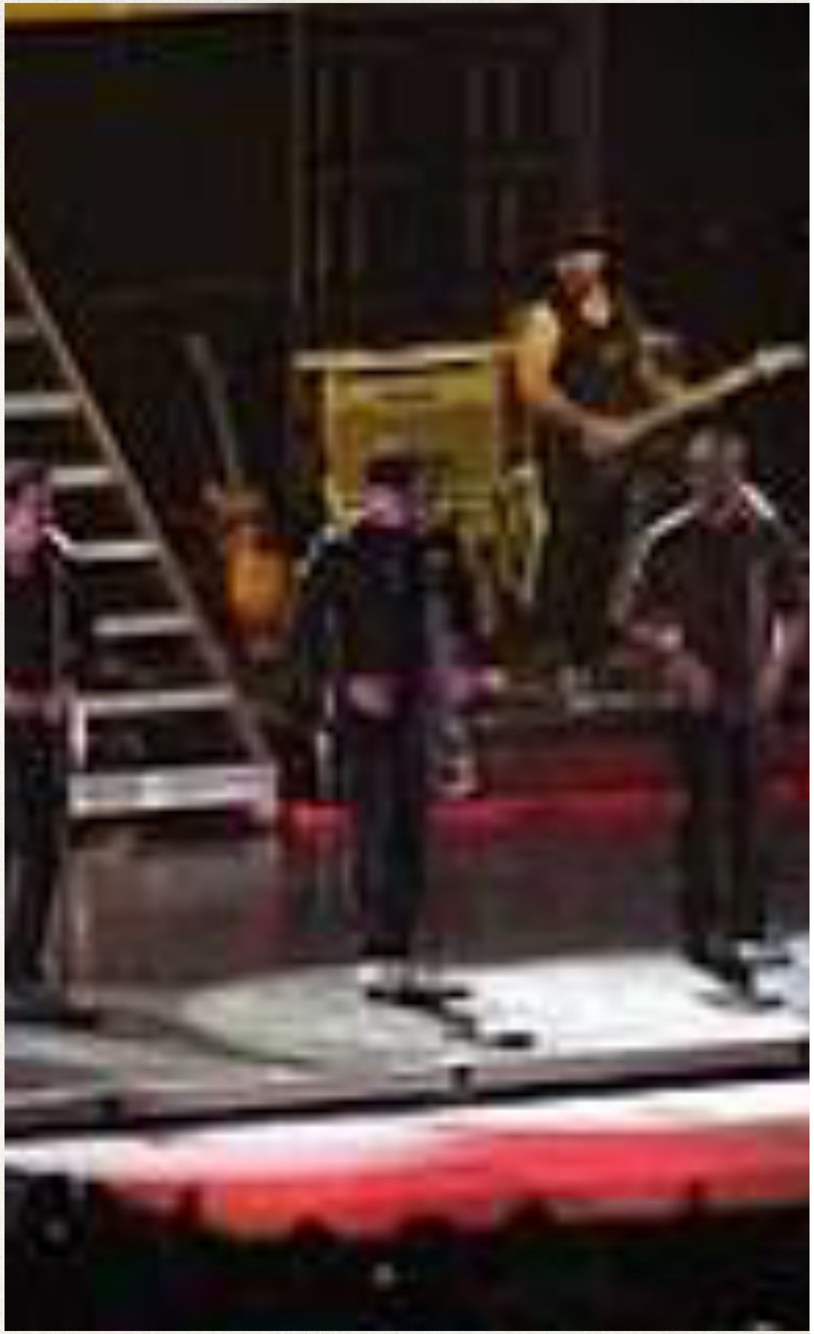
(photo: Palace of Knossos, Crete)



Univalent foundations, a new kid on the block

Do they replace all we have known?

Photo: New kids on the block, a music group from Boston



« It’s impossible to overstate the consequences for philosophy, especially the philosophy of Mathematics, if Voevodsky’s proposed new Foundations were adopted. »

Michael Harris, Mathematics without apologies (2015)

Set theory is based on classical first order logic. Basic entities are sets and they are completely determined by their elements:

Axiom of extensionality: for any A and B,

$$[\forall x(x \in A \iff x \in B)] \iff A = B.$$

The law of excluded middle holds and the axiom of Choice is assumed in ZFC.

Univalent foundations are a vision of Vladimir Voevodsky of a foundational system for mathematics

The basic objects are homotopy types.

They obey a type theory satisfying the univalence axiom.

They are formalisable in a computer proof assistant and are constructivist.

In set theory the basic variables are sets and the atomic formulas are of the form $x \in y$ and $x = y$

Type theory goes back to Russell, who used the idea of a type to resolve the Russell paradox. The idea is simple: basic objects are not just sets connected with membership, but each set has a Type. There are several versions of TT.

Each variable in TT comes with a level, which is a natural number

x^i means that x is a variable of type i.

Basic formulas are of the form

$$x^i \in y^{i+1} \text{ and } x^i = z^i$$

Adding the axiom of infinity to level 0 and the axiom of choice at every level leads a system (equivalent to the system of) Principia Mathematicae by Russell and Whitehead. It can derive all mathematics known at the time of PM but is cumbersome: think of what a complement or a cardinal number means in this context.

But for HOTT it is more useful to think of types as coming from programming, starting with Dependent Type Theory, à là Martin-Löf.

Here we have expressions such as **term:Type** (called **contexts**)

$$3x\cdot 2+1,\mathrm{inr}(3/4-y)) : \mathbb{N}\times(\mathbb{R}+\mathbb{Q})$$

The above only makes sense if $x:\mathbb{N}$ and $y:\mathbb{Q}$, so we might like to specify it with a **judgement (proof)**:

$$x:\mathbb{N},\ y:\mathbb{Q}\ \vdash 3x\cdot 2+1,\mathrm{inr}(3/4-y)) : \mathbb{N}\times(\mathbb{R}+\mathbb{Q})$$

It is useful to think of this as a category, in which objects are the contexts and morphisms are the judgements. **Classifying category, Ctx.**

Example: $\Gamma \vdash a : A$

means that a is a morphism from the context Γ to the context A.

We endow the contexts with natural operations, such as the product:

$x{:}A \times y{:}B$ can be understood as $\{x{:}A, y{:}B\}$

Contexts with variables are "types" and those where we substituted a variable by a constant are morphisms.

To make this into dependant type theory, we handle the idea of substitution.

We allow "families of contexts" such as "B **type**", where we interpret

"$\Gamma \vdash B$ **type**"

as a judgement allowing to conclude from Γ that some object is of type B. Intuition: B "depends on " Γ.

We can also combine dependant types using a join:

<u>Example</u>: "$n : N+ \vdash C_n$ **type**" corresponds to

"for all n, we have an object of type C_n"

We can think of this as of "slice categories" , for example

"$\Gamma \vdash B$ **type**" corresponds to the category Ctx/Γ, all contexts in which Γ holds.

The point: substitution into a dependent type presents the pullback functor between slice categories

Now we can make a proof system out of this by allowing deductions, which are the axioms of the judgements.

These axioms come in two kinds: ones that allow us to construct a type and others which allow to eliminate in order to prove equivalence.

This resembles to, for example, the natural deduction, but in this context is more complex.

A typical rule is of the form:

$$\frac{\Gamma \vdash A \ \mathbf{type}\ ; \Gamma, x : A \vdash B \ \mathbf{type}}{\Gamma \vdash \Pi_{n:A} B \ \mathbf{type}}$$

This type of reasoning is well suited for implementation on a computer, hence proof assistants such as Coq.

Choosing a particular collection of type constructors specifies the rules of a particular type theory.

It is possible to interpret propositional logic in type theory, namely we interpret propositions P as types with at most one element. P is “true“ if there is an element in it (these are types of level 1). We can also interpret sets, as properties (types of level 2).

By a property of elements of a type A we mean a judgment

$$x{:}A \vdash P{:}\Omega.$$

Under this interpretation, we obtain the Curry-Howard correspondance between the operations of **intuitionistic** propositional logic and (some of) the constructors of type theory. So theorems of **intuitionistic** propositional logic are provable in the deduction system.

The Law of Excluded Middle does not fall into this correspondance. LEM is addable artificially as a judgement, but then it causes an infinite loop in the execution of the proof assistants.

Note: Axiom of Choice is contradictory to the intuitionistic logic. In a model of DTT it can be enforced on level of the sets, at the price of losing the connection with the proof assistants.

A problem with identity

« Help! Who Am I? 7 Signs That You Suffer From an Identity Crisis

By Harley Therapy »

Frege: *“Identity is a relation given to us in such a specific form that it is inconceivable that various kinds of it should occur “*

- Yet in type theory, the context gives rise to two different kinds of identity: *definitional* “A=B” (for types) and “x = y : A” for elements of a type, and the *propositional* one $Id_A(x,y)$ for propositions x,y which stands that x and y are provably equivalent propositions (by Curry-Howard correspondence). Then we can also declare that any two proofs for this identity are equivalent etc., extending the propositional equivalence to all levels of types.
- Definitional = implies the propositional =, but the other way around?
- Martin-Lof and others tried to resolve this by declaring it specifically but Girard (1972) found that this leads to a paradox similar to the Russell’s paradox

The topological view

Streicher in 1993 almost got a model of TT where the two versions of = were equal, by modelling identity types of the 1st level like like $Id_A(x, y)$ by abstract grupoids. In this approach the propositional identity of all higher types is actually definitional.

The problem of two identities also appeared in homotopy theory and was known to Voevodsky, who knew that it could not be solved "internally" but needed an introduction of an ∞ - grupoid.

Key idea: interpret the category Ctx as the homotopy category. Types become spaces and morphisms. Propositional = becomes homotopic equivalence. The structure obtained is an ∞ - grupoid.

$\boxed{\text{Univalence axiom}}$

The definitional equivalence of two types A=B
is propositionally
equivalent to their propositional equivalence,
written $A \approx B$

Δ is the simplex category, whose objects are linear orders on a natural number {0,1,2 …, n-1} and morphisms are (non-strictly) order-preserving functions.

"small sets" are sets smaller than some fixed cardinal (maybe a large cardinal). Their category is called **Set.**

A simplicial set X is a contravariant functor X: $\Delta \to$ Set.

Simplicial sets form a category, under "natural transformations" denoted by sSet.

A certain fibration sSeT/W of this category is equivalent to the category of "nice" (only one homotopy group) topological spaces and homotopy classes of maps. This was discovered by Ellenberg and MacLane in 1953.

Because of this fact one can, when working up to homotopy, think of simplicial sets as of combinatorial representations of shapes or spaces, of simplicial paths as paths on these spaces etc.

Theorem (Voevoydsky 2010) Modulo the existence of two inaccessible cardinals, it is consistent that sSet/W forms a model of the univalence axiom.

Furthemore, we can assume LEM on the level of propositions and AC on the level of sets.

Voyevodsky developed a Coq computer library of proofs which goes with simplical sets, UniLab (he was later helped by many others).

The ZFC set theory will remain the most important benchmark of consistency.

Voevodsky 2013 (Logic Colloquium)

This all fits into a vision from algebraic topology developed by Groethendick in “Pursuing Stacks” in 1983.

Homotopy Hypothesis : Ellenberg-Mac Lane correspondance holds between all topological spaces and an infty-grupoid.

This requires higher category theory and is still open.

Kapranov and Voyevodsky (1990) had a false proof of the version of this saying that for every n there is an n-version of Ellenberg-Mac Lane.

Important Note: Recent work of **Aczel (2017)** , shows that the proof theoretic strength of Martin-Lof/dependent type theory is much less than that of ZFC. In his model we can only interpret constructive sets.

This model can be done in constructive set theory with universes. This set theory is proof theoretically strictly weaker than than ZFC, and has the same strength as Martin-Lof/dependent type theory (without the univalence axiom). A corollary of the work by Bezem, Coquand and Huber on the cubical sets model is that adding the univalence axiom does not add any proof theoretic power to dependent type theory.

On the other hand, work of Crosilla and Rathjen (2002) shows that the consistency strength of CFZ+ LEM+universes is at least ZFC+ unboundedly many inaccessibles

Conclusion: It is LEM that adds strength to univalent foundations. Univalent foundations without LEM cannot resolve questions of high consistency strength.

Constructive arguments in mathematics can be expressed in the univalent foundations and formalised.

Univalent foundations

Vladimir Voevodsky's vision of a foundational system
for mathematics in which :
(1) the basic objects are homotopy types,
(2) it is based on a type theory satisfying the univalence axiom,
and
(3) it is formalized in a computer proof assistant.

Staying close to reality

- Some think that the art of mathematics is to deal with constructive, calculable and definable objects (Kronecker, Poincaré to some extent) and Brouwer (to some extent). They reject the idea of the “actual infinity”

- Recent results in foundations of mathematics suggest that constructive, calculable and definable mathematics is doable by a machine, using computer libraries rather than any art.

- What do we conclude about the “actual infinity” then? To me it seems that the “actual infinity” and its dependence on the proofs by contradiction and LEM is what remains crucial in the art of mathematics, like it or not !

What about foundations?

« On sait aujourd'hui qu'il est possible, logiquement parlant, de faire *presque* dériver toute les mathématiques actuelles d'une source unique, la théorie des ensembles... Ce faisant, nous ne prétendons pas légiférer pour l'éternité ; il se peut qu'un jour les mathématiciens s'accordent à se permettre des modes de raisonnement non formalisables dans le langage exposé ici; suivant certains, l'évolution récente des théories d'homologie dites axiomatiques donnerait à penser que ce jour n'est pas si éloigné. Il faudrait alors sinon changer complètement de langage, tout au moins élargir les règles de la syntaxe. C'est à l'avenir qu'il appartiendra d'en décider. »

-Bourbaki, Introduction to Book I, foundations 1966

Representations

Working mathematics is a representation of the real mathematics

(*hard-core platonist*)

``Mathematics is a whole; but it has been known since the early 20th century that some formal systems, such as Zermelo–Fraenkel set theory, can encode almost all of mathematics. (..) by a "universe of mathematics" I mean a model of a formal system in which mathematics can be encoded.“

(*Michael Schulman, HOTT book*)

Various universes may satisfy different foundational axioms.

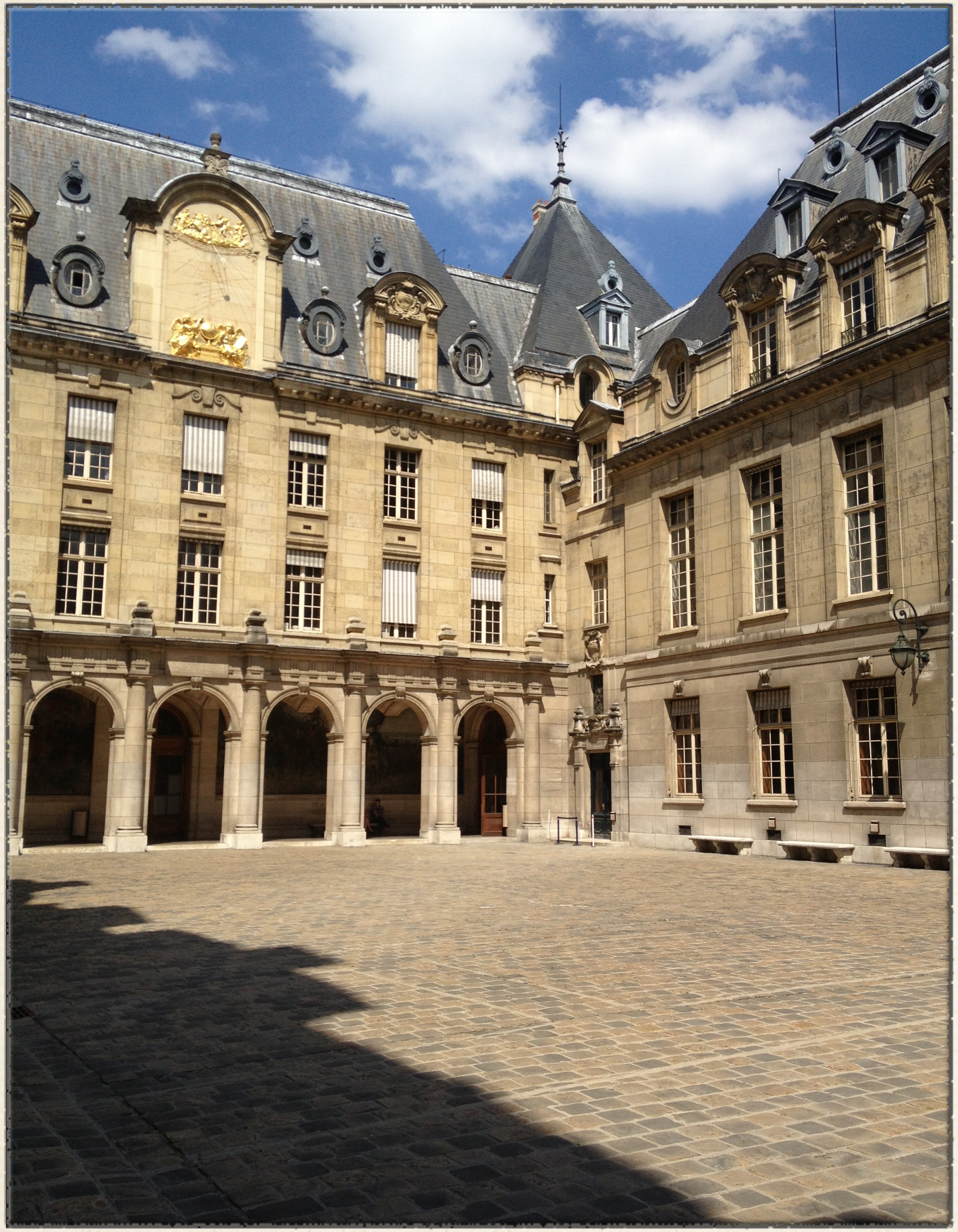
- Set theory has its place in the foundations of mathematics because it gives foundations for an important part of mathematics in a way that is consistent with the usual practice.

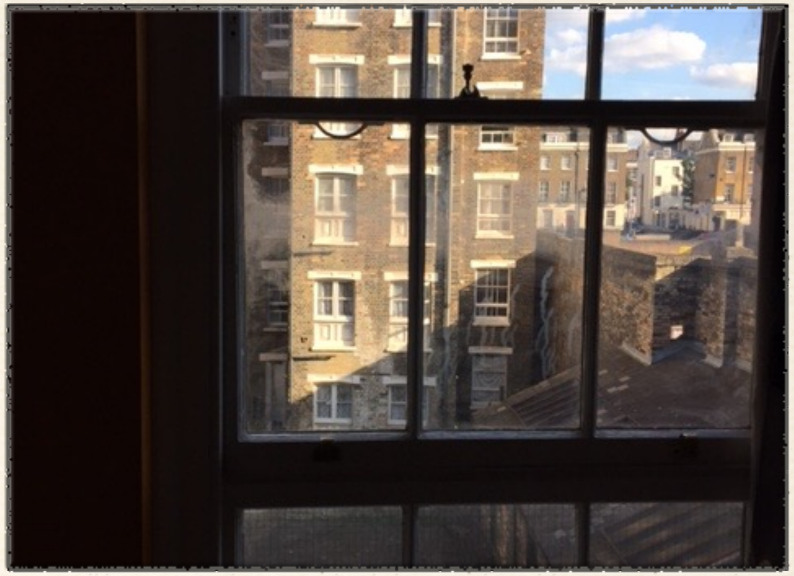
- Category theory has its place since it provides a way to model parts of mathematics that depend on proper classes, such as algebraic topology.

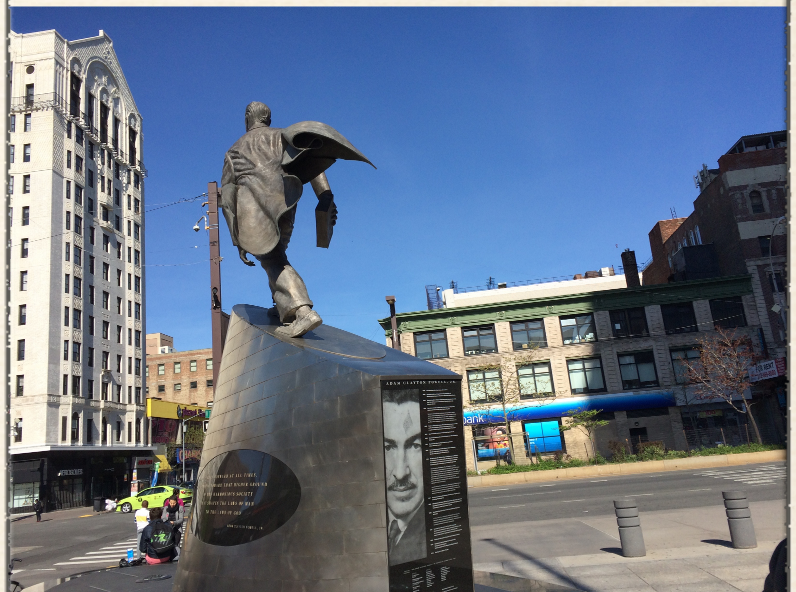
- Univalent foundations have their place because they give us a new way to discuss proofs, a central notion everywhere in mathematics.

This approach represents a shift in the use of the word ‘foundation’ from the ontological view of the 20th century philosophers of mathematics who were looking for the truth or the essence of mathematics.

The more recent view is more epistemic, since the practicing mathematician is more after understanding than after ‘knowledge’.



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